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A Faddeev Sequence Method for Solving Lyapunov and Sylvester Equations

Bernard Hanzon*

Department of Econometrics
Free University Amsterdam
De Boelelaan 1105
1081 HV Amsterdam, The Netherlands

and

Ralf L. M. Peeters†

Department of Mathematics
University of Limburg
P.O. Box 616
6200 MD Maastricht, The Netherlands

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ABSTRACT

Lyapunov and Sylvester equations play an important role in linear systems theory. This paper deals with a method of solving such equations of the form $AP + PB = K$ and $P - APB = K$ with $A \in \mathbf{R}^{m\times m}$, $B \in \mathbf{R}^{n\times n}$, and P, $K \in \mathbf{R}^{m\times n}$, by exploiting the matrix-algebra structure of the problem. No use is made of Kronecker products and the largest matrices occurring in the algorithms are of sizes $m \times m$, $n \times n$, and $m \times n$. The Faddeev method for matrix inversion lies at the very heart of the algorithms presented. It occurs on several levels of the problem: for the matrices A and B and for the Lyapunov and Sylvester operators. The resulting algorithms are capable of solving the equations in a finite number of recursion steps. They are very much apt for symbolic calculation. It is shown how a solution can be quickly obtained for an equation with an arbitrary right-hand side K, provided a solution is known for a right-hand side xy^T of rank 1, where (A, x) and (B^T, y) are reachable pairs. The concept of a Faddeev reachability matrix introduced here turns out to be very useful. It establishes a close connection between the controller canonical (companion) form of a reachable pair (A, b) and the Faddeev sequence of A. If A is already on

*E-mail: bhanzon@econ.vu.nl.
†E-mail: ralf.peeters@math.rulimburg.nl.

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controller form, then its Faddeev sequence takes on an especially simple form. Also in the symmetric case where $A = B^T$, many important simplifications arise. For this case alternative algorithms that require less iterations are developed. The paper concludes with some examples concerning the symbolic solution of the Lyapunov equation $AP + PA^T = bb^T$ with (A, b) on controller form, showing the potential of the algorithms.

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1. INTRODUCTION

In the theory of linear dynamical systems, linear matrix equations like Lyapunov and more generally Sylvester equations play an important role. There are many numerically well-tested algorithms for such equations. However, for several applications it is important to obtain symbolic solutions of such equations. For example, in model reduction theory one encounters optimization problems in which the criterion function can be expressed in terms of the solutions of one or several Lyapunov equations. In order to apply gradient search algorithms, one needs the derivatives of the criterion function. Also if one wants to solve the first-order conditions algebraically the derivatives of the criterion function are required. Therefore having an explicit symbolic expression of the criterion function is very useful. Also in the application of techniques from Riemannian geometry to problems in systems theory, like system identification, model reduction, parametrization, the calculation of Riemannian metric tensors plays an important role and often involves the solution of several Lyapunov and Sylvester equations. Again in that case it is very useful to have the answers in symbolic form, because further calculations to obtain curvature tensors, etc., require taking derivatives. For stochastic linear dynamical models the Fisher information matrix is in fact a Riemannian metric tensor and it can also be obtained in symbolic form by solving a number of Lyapunov and Sylvester equations. For further information on these issues the reader is referred to [4, 5, 10].

One straightforward approach to solving such equations symbolically is to use the fact that the equations are linear and use Kronecker products to transform the problem into one in which an $mn \times mn$ matrix has to be inverted, if the matrix sought is $m \times n$. This usually works in practice, i.e., with a computer algebra package like Maple[1] or Mathematica,[2] *only* for problems in which the values of m and n are small. In this paper an alternative approach, which exploits the matrix algebra structure of the problem is presented. In existing algorithms of this kind (cf. [3, 7]) one

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[1] Maple is a registered trademark of Waterloo Maple Software.

[2] Mathematica is a registered trademark of Wolfram Research, Inc.

needs to perform first a number of polynomial calculations starting with the characteristic polynomial. Here a different, but theoretically related approach in which calculations with the characteristic polynomial are avoided is taken. The idea behind the presented algorithm is to apply Faddeev's algorithm for inversion of a linear finite-dimensional operator to Lyapunov and Sylvester equations. This leads to a recursive procedure that ends after a finite number of steps, related to the size of the problem. In the algorithms the use of Kronecker products is avoided and the largest matrices occurring are of sizes $n \times n, m \times m, m \times n$. Due to the recursive structure of the algorithm, it can be programmed in a concise way.

The well-known observability and reachability Grammians of a stable linear dynamical time invariant state space system are solutions of Lyapunov equations. A linear dynamical time invariant state space system is called a linear system, to remain concise. An important special case of our method occurs if a single-input single-output stable linear system is in *controller canonical form*. This form is not to be confused with the *controllability canonical form*. It turns out that the controller canonical form can in fact be understood as the canonical form that is obtained by choosing the basis of the state space in a way that is directly related to the Faddeev sequence of the dynamical matrix of the system. Although the controller canonical form is one of the most well-known canonical forms, to the best of our knowledge this has not been noted before. In the controller canonical form the dynamical matrix is in companion form. It turns out that the Lyapunov equation can be reduced to the special case in which the dynamical matrix is in companion form. We give the symbolic solution to the Lyapunov equation for this case for a number of choices for n.

Some experiences with the algorithm in computer algebra calculations are reported upon. We encountered cases that could not be handled by a direct Kronecker products method, because of memory problems, which could be handled by the method proposed here. Concerning the number of calculations involved, at least for numerical calculations this method appears to be quite efficient; however, experience shows that it is numerically unreliable. Of course this does *not* play a role in computer algebra applications where exact arithmetic is used.

2. THE FADDEEV SEQUENCE OF A MATRIX AND MATRIX INVERSION

One of the basic problems in linear algebra is the calculation of the inverse of a square $n \times n$ nonsingular matrix A. An interesting matrix-algebra method with which to calculate the inverse can be obtained by exploiting the properties of the Faddeev sequence of A. The Faddeev sequence of the matrix A is recursively defined as follows:

$$\begin{aligned} A(0) &:= I_n \\ \widetilde{A}(0) &:= A(0)A \end{aligned} \tag{2.1}$$

$$\begin{aligned} A(k) &:= \widetilde{A}(k-1) - \frac{\operatorname{tr}(\widetilde{A}(k-1))}{k} I_n, \qquad k = 1, 2, \ldots \\ \widetilde{A}(k) &:= A(k)A. \end{aligned} \tag{2.2}$$

Let the characteristic polynomial of A be given by $p(s) = \det(sI_n - A) = s^n + p_1 s^{n-1} + \cdots + p_n$. Then the Faddeev sequence has the following nice properties, derived from the Newton identities (cf. [2, p. 87]):

$$p_k = -\frac{\operatorname{tr}(\widetilde{A}(k-1))}{k}, \qquad k = 1, 2, \ldots, n \tag{2.3}$$

and

$$A(k) = A^k + p_1 A^{k-1} + p_2 A^{k-2} + \cdots + p_{k-1} A + p_k I_n, \qquad k = 0, 1, 2, \ldots, n. \tag{2.4}$$

So due to the theorem of Cayley–Hamilton $A(n) = p(A) = 0$. (From this it follows directly that $A(k) = 0$ for all $k \geq n$.) Therefore if A is nonsingular,

$$-\frac{\operatorname{tr}(\widetilde{A}(n-1))}{n} = p_n = \det(-A) \neq 0$$

and

$$0 = A(n) = \widetilde{A}(n-1) - \frac{\operatorname{tr}(\widetilde{A}(n-1))}{n} I_n$$

from which the following formula for the inverse of A can be derived easily:

$$A^{-1} = \frac{n}{\operatorname{tr}(\widetilde{A}(n-1))} A(n-1). \tag{2.5}$$

Note that the inverse is obtained by a sequence of operations consisting of multiplication by A, taking trace, subtraction of a scalar multiple of the identity matrix, and division by a scalar. Therefore the algorithm can in fact be applied to any finite-dimensional linear endomorphism (i.e., linear operator of which the domain is equal to the codomain) and is independent of the choice of basis in the vector space in which the operator acts. Therefore one can speak of the Faddeev sequence of a linear endomorphism and the inverse of a linear endomorphism can be constructed from the Faddeev sequence in the way described. This will be important in the following sections.

It may be good to note at this point that in the literature the Faddeev sequence is usually presented as a means to calculate the resolvent $(sI_n - A)^{-1}$ of a matrix A. The relevant formula for the resolvent is

$$(sI_n - A)^{-1} = \frac{A(0)s^{n-1} + A(1)s^{n-2} + \cdots + A(n-2)s + A(n-1)}{p(s)},$$

where $p(s)$ also follows from the Faddeev sequence calculations as noted before. (Compare, e.g., [1, Sect. 3.4, 3.5].)

3. SOLVING LYAPUNOV AND SYLVESTER EQUATIONS USING FADDEEV SEQUENCES

3.1. *The Matrix-Algebra Approach to Lyapunov and Sylvester Equations*

Consider Sylvester equations of the form

$$AP + PB = K \tag{3.1}$$

and

$$P - APB = K, \tag{3.2}$$

where A is a given $m \times m$ matrix, B is a given $n \times n$ matrix, K is a given $m \times n$ matrix, and P is an unknown $m \times n$ matrix for which we want to solve the equation. To do that consider the linear matrix operators $\mathcal{L} = \mathcal{L}_A$, $\mathcal{R} = \mathcal{R}_B$, and $\mathcal{I}_{mn}$ defined by

$$\mathcal{L}_A\colon\ P \mapsto AP$$

$$\mathcal{R}_B\colon\ P \mapsto PB$$

and

$$\mathcal{I}_{mn}\colon\ P \mapsto P.$$

This last one is clearly the identity on the vector space of $m \times n$ matrices. Define the linear operators $\mathcal{C} := \mathcal{L} + \mathcal{R}$ and $\mathcal{D} := \mathcal{I}_{mn} + \mathcal{L}\mathcal{R}$. Then the Sylvester equations (3.1) and (3.2) can be written as

$$\mathcal{C}(P) = K \tag{3.3}$$

and

$$\mathcal{D}(P) = K \tag{3.4}$$

respectively. Because $\mathcal{C}$ and $\mathcal{D}$ are linear endomorphisms (on the vector space of $m \times n$ matrices), the solution is given abstractly by

$$P = \mathcal{C}^{-1}(K)$$

and

$$P = \mathcal{D}^{-1}(K),$$

respectively, if $\mathcal{C}$ and $\mathcal{D}$ are invertible. This is known to be the case iff A and $-B$ have no eigenvalues in common and iff no eigenvalue of A is the reciprocal of an eigenvalue of B, respectively (cf., e.g., [2]). Therefore the question arises how $\mathcal{C}^{-1}$ and $\mathcal{D}^{-1}$ can be calculated. In principle the techniques of the previous section can be applied: One can define the Faddeev sequence $\{\mathcal{C}(k), \widetilde{\mathcal{C}}(k) \mid k = 0, 1, 2, ...\}$, for which $\mathcal{C}(k) = 0$, $\widetilde{\mathcal{C}}(k) = 0$ for all $k \geq nm$. Then $\mathcal{C}^{-1} = (mn/\text{tr}\ \widetilde{\mathcal{C}}(mn-1))\mathcal{C}(mn-1)$ and similarly for $\mathcal{D}$. To apply this idea one needs an explicit representation of $\mathcal{C}$ and its Faddeev sequence. Here we propose to represent $\mathcal{C}$ and the elements of its Faddeev sequence as a linear combination of operators $\mathcal{L}_i\mathcal{R}_j$, where $\mathcal{L}_i := \mathcal{L}_{A(i)}$ denotes left multiplication of an $m \times n$ matrix by the matrix $A(i)$ from the Faddeev sequence of A and where similarly $\mathcal{R}_j := \mathcal{R}_{B(j)}$ denotes right multiplication of an $m \times n$ matrix by the matrix $B(j)$ from the Faddeev sequence of B. The reason why $\mathcal{C}$ and the elements of its Faddeev sequence can be represented in this way is twofold:

(i) Left multiplication of an $m \times n$ matrix by a matrix $\overline{A}$ and right multiplication of an $m \times n$ matrix by an $n \times n$ matrix $\overline{B}$ are commutative (due to the associativity of matrix multiplication): $\mathcal{L}_{\overline{A}}\mathcal{R}_{\overline{B}} = \mathcal{R}_{\overline{B}}\mathcal{L}_{\overline{A}}$ for all $\overline{A}$ and $\overline{B}$ of the correct sizes.

(ii) An arbitrary polynomial in A can be written as a linear combination of elements $A(i)$, $i = 0, 1, \ldots$ of the Faddeev sequence of A, because $A(i) = A^i +$ lower degree terms for each $i = 0, 1, 2, \ldots$. Because $A(i) = 0$ for all $i \geq m$, this implies that it can actually be written as a linear combination of $A(0), A(1), \ldots, A(m-1)$. It follows directly that an arbitrary polynomial in $\mathcal{L}$ can be written as a linear combination of $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{m-1}$. Similarly an arbitrary polynomial in $\mathcal{R}$ can be written as a linear combination of $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{n-1}$.

Combining (i) and (ii) one finds that each polynomial in $\mathcal{C}$ can be rewritten as a linear combination of $\{\mathcal{L}_i\mathcal{R}_j \mid i = 0, 1, 2, \ldots, m-1;\ j = 1, 2, \ldots, n-1\}$. Note that the linear combination does not necessarily have to be unique. (It is not iff the degree of the minimal polynomial of $\mathcal{C}$ is less than the degree of its characteristic polynomial.) Even if there is linear dependence of this kind, the algorithm presented below still works without any changes, although a more careful analysis may then produce a quicker algorithm. The symmetric case $B = A^T$ is an important example of such a situation, to which we return in Section 5.

3.2. Derivation of the Faddeev Sequence Formulae

An important ingredient in the calculation of the Faddeev sequences of $\mathcal{C}$ and $\mathcal{D}$ is the calculation of the traces of $\widetilde{\mathcal{C}}(k)$ and $\widetilde{\mathcal{D}}(k)$. Because $\widetilde{\mathcal{C}}(k)$ is represented as a linear combination of endomorphisms of the form $\mathcal{L}_i\mathcal{R}_j$ the question arises how one can calculate the trace of $\mathcal{L}_i\mathcal{R}_j$. The answer is given in the following lemma.

LEMMA 3.1.

$$\operatorname{tr}\{\mathcal{L}_i\mathcal{R}_j\} = \operatorname{tr} A(i) \operatorname{tr} B(j). \tag{3.5}$$

Proof. For each $i \in \{0,\dots,m-1\}$, $j \in \{0,\dots,n-1\}$, $\mathcal{L}_i\mathcal{R}_j$ is a linear endomorphism of the vector space of $m \times n$ matrices. Therefore its trace is well defined, independent of the specific choice of a basis in the vector space. Consider the inner product $\langle\cdot,\cdot\rangle$ on the vector space $\mathbf{R}^{m\times n}$ of $m \times n$ matrices given by

$$\langle P, Q\rangle = \operatorname{tr}\{P^TQ\}; \qquad P, Q \in \mathbf{R}^{m\times n}.$$

Then an orthogonal basis is given by $\{E_{kl} = e_kf_l^T \mid k = 1,\dots,m; l = 1,\dots,n\}$ with e_k the kth standard basis vector in $\mathbf{R}^m$ and f_l the lth standard basis vector in $\mathbf{R}^n$. The trace of an endomorphism $\mathcal{L}_i\mathcal{R}_j$ of $\mathbf{R}^{m\times n}$ is equal to

$$\begin{aligned}\sum_{k=1}^{m}\sum_{l=1}^{n}\langle E_{kl}, \mathcal{L}_i\mathcal{R}_j(E_{kl})\rangle &= \sum_{k=1}^{m}\sum_{l=1}^{n}\operatorname{tr}\{\left(e_kf_l^T\right)^T A(i)e_kf_l^TB(j)\}\\ &= \sum_{k=1}^{m}\sum_{l=1}^{n} e_k^TA(i)e_kf_l^TB(j)f_l\\ &= \left(\sum_{k=1}^{m} e_k^TA(i)e_k\right)\left(\sum_{l=1}^{n} f_l^TB(j)f_l\right)\\ &= \operatorname{tr} A(i)\operatorname{tr} B(j).\end{aligned}$$

■

THEOREM 3.2. *Define the $m \times n$ coefficient matrices*

$$C(k) = (c_{ij}(k))_{i=0,j=0}^{m-1,n-1}, \qquad k = 0,1,\dots,mn-1 \tag{3.6}$$

$$\widetilde{C}(k) = (\widetilde{c}_{ij}(k))_{i=0,j=0}^{m-1,n-1}, \qquad k = 0,1,\dots,mn-1 \tag{3.7}$$

by the following recursive formulas

$$C(0) = E_{11} \tag{3.8}$$

and for each $k = 0, 1, 2, \ldots, mn-1$

$$\widetilde{c}_{00}(k) = \sum_{i=0}^{m-1} \frac{\operatorname{tr} \widetilde{A}(i)}{i+1} c_{i0}(k) + \sum_{j=0}^{n-1} c_{0j}(k) \frac{\operatorname{tr} \widetilde{B}(j)}{j+1} \tag{3.9}$$

$$\widetilde{c}_{0j}(k) = \sum_{i=0}^{m-1} \frac{\operatorname{tr} \widetilde{A}(i)}{i+1} c_{ij}(k) + c_{0,j-1}(k) \qquad \textit{if } j > 0 \tag{3.10}$$

$$\widetilde{c}_{i0}(k) = \sum_{j=0}^{n-1} c_{ij}(k) \frac{\operatorname{tr} \widetilde{B}(j)}{j+1} + c_{i-1,0}(k) \qquad \textit{if } i > 0 \tag{3.11}$$

$$\widetilde{c}_{ij}(k) = c_{i-1,j}(k) + c_{i,j-1}(k) \qquad \textit{if } i > 0 \textit{ and } j > 0 \tag{3.12}$$

$$c_{00}(k+1) = \widetilde{c}_{00}(k) - \frac{1}{k+1} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \widetilde{c}_{ij}(k) \operatorname{tr} A(i) \operatorname{tr} B(j) \tag{3.13}$$

$$c_{ij}(k+1) = \widetilde{c}_{ij}(k) \qquad \textit{if } (i,j) \neq (0,0). \tag{3.14}$$

Then for each $k = 0, 1, 2, \ldots, mn-1$

$$\mathcal{C}(k) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} c_{ij}(k) \mathcal{L}_i \mathcal{R}_j \tag{3.15}$$

$$\widetilde{\mathcal{C}}(k) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \widetilde{c}_{ij}(k) \mathcal{L}_i \mathcal{R}_j. \tag{3.16}$$

Proof. Induction is used for $k = 0, 1, 2, \ldots, mn-1$. Clearly

$$\mathcal{C}(0) = \mathcal{I}_{mn} = \mathcal{L}_0 \mathcal{R}_0 = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} c_{ij}(0) \mathcal{L}_i \mathcal{R}_j$$

with $c_{00}(0) = 1$ and $c_{ij}(0) = 0$ if $(i,j) \neq (0,0)$. Now suppose (this is the induction hypothesis) that

$$\mathcal{C}(k) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} c_{ij}(k) \mathcal{L}_i \mathcal{R}_j;$$

then

$$\widetilde{\mathcal{C}}(k) = \mathcal{C}\mathcal{C}(k) = (\mathcal{L} + \mathcal{R}) \sum_{ij} c_{ij}(k) \mathcal{L}_i \mathcal{R}_j = \sum_{ij} c_{ij}(k) (\mathcal{L}\mathcal{L}_i \mathcal{R}_j + \mathcal{L}_i \mathcal{R}\mathcal{R}_j).$$

Applying $\mathcal{L}\mathcal{L}_i$ to an $m \times n$ matrix means multiplication on the left by

$$AA(i) = A(i)A = \widetilde{A}(i) = A(i+1) + \frac{\operatorname{tr}\widetilde{A}(i)}{i+1} I_m = A(i+1) + \frac{\operatorname{tr}\widetilde{A}(i)}{i+1} A(0)$$

by definition of $A(i+1)$ and $A(0)$. Therefore

$$\mathcal{L}\mathcal{L}_i = \mathcal{L}_{i+1} + \frac{\operatorname{tr}\widetilde{A}(i)}{i+1}\mathcal{L}_0.$$

Similarly

$$\mathcal{R}\mathcal{R}_j = \mathcal{R}_{j+1} + \frac{\operatorname{tr}\widetilde{B}(j)}{j+1}\mathcal{R}_0.$$

Because $\mathcal{L}_m = \mathcal{R}_n = 0$ it follows that

$$\begin{aligned}
\widetilde{C}(k) &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{ij}(k)\left(\mathcal{L}_{i+1}\mathcal{R}_j + \frac{\operatorname{tr}\widetilde{A}(i)}{i+1}\mathcal{L}_0\mathcal{R}_j + \mathcal{L}_i\mathcal{R}_{j+1} + \frac{\operatorname{tr}\widetilde{B}(j)}{j+1}\mathcal{L}_i\mathcal{R}_0\right) \\
&= \sum_{i=0}^{m-2}\sum_{j=1}^{n-1} c_{ij}(k)\mathcal{L}_{i+1}\mathcal{R}_j + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{ij}(k)\frac{\operatorname{tr}\widetilde{A}(i)}{i+1}\mathcal{L}_0\mathcal{R}_j \\
&\quad + \sum_{i=0}^{m-2} c_{i,0}(k)\mathcal{L}_{i+1}\mathcal{R}_0 + \sum_{i=1}^{m-1}\sum_{j=0}^{n-2} c_{ij}(k)\mathcal{L}_i\mathcal{R}_{j+1} \\
&\quad + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{ij}(k)\frac{\operatorname{tr}\widetilde{B}(j)}{j+1}\mathcal{L}_i\mathcal{R}_0 + \sum_{j=0}^{n-2} c_{0,j}(k)\mathcal{L}_0\mathcal{R}_{j+1} \\
&= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} (c_{i-1,j}(k) + c_{i,j-1}(k))\mathcal{L}_i\mathcal{R}_j + \sum_{j=0}^{n-1}\left(\sum_{i=0}^{m-1}\frac{\operatorname{tr}\widetilde{A}(i)}{i+1}c_{ij}(k)\right)\mathcal{L}_0\mathcal{R}_j \\
&\quad + \sum_{i=0}^{m-1}\left(\sum_{j=0}^{n-1} c_{ij}(k)\frac{\operatorname{tr}\widetilde{B}(j)}{j+1}\right)\mathcal{L}_i\mathcal{R}_0 + \sum_{i=1}^{m-1} c_{i-1,0}(k)\mathcal{L}_i\mathcal{R}_0 \\
&\quad + \sum_{j=1}^{n-1} c_{0,j-1}(k)\mathcal{L}_0\mathcal{R}_j.
\end{aligned}$$

This is equal to $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\widetilde{c}_{ij}(k)\mathcal{L}_i\mathcal{R}_j$ if the $\widetilde{c}_{ij}(k)$ are assigned to be

$$\widetilde{c}_{00}(k) = \sum_{i=0}^{m-1}\frac{\operatorname{tr}\widetilde{A}(i)}{i+1}c_{i0}(k) + \sum_{j=0}^{n-1} c_{0j}(k)\frac{\operatorname{tr}\widetilde{B}(j)}{j+1}$$

$$\widetilde{c}_{0j}(k) = \sum_{i=0}^{m-1} \frac{\operatorname{tr} \widetilde{A}(i)}{i+1} c_{ij}(k) + c_{0,j-1}(k) \qquad \text{if } j > 0$$

$$\widetilde{c}_{i0}(k) = \sum_{j=0}^{n-1} c_{ij}(k) \frac{\operatorname{tr} \widetilde{B}(j)}{j+1} + c_{i-1,0}(k) \qquad \text{if } i > 0$$

$$\widetilde{c}_{ij}(k) = c_{i-1,j}(k) + c_{i,j-1}(k) \qquad \text{if } i > 0 \text{ and } j > 0.$$

Next consider the equation

$$\mathcal{C}(k+1) = \widetilde{\mathcal{C}}(k) - \frac{\operatorname{tr} \widetilde{\mathcal{C}}(k)}{k+1} \mathcal{I}_{mn}.$$

First note that application of Lemma 3.1 gives

$$\begin{aligned} \frac{\operatorname{tr} \widetilde{\mathcal{C}}(k)}{k+1} &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \widetilde{c}_{ij}(k) \frac{1}{k+1} \operatorname{tr} (\mathcal{L}_i \mathcal{R}_j) \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \widetilde{c}_{ij}(k) \frac{1}{k+1} \operatorname{tr} A(i) \operatorname{tr} B(j). \end{aligned}$$

It follows that

$$\mathcal{C}(k+1) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \widetilde{c}_{ij}(k) \mathcal{L}_i \mathcal{R}_j - \left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \frac{\widetilde{c}_{ij}(k)}{k+1} \operatorname{tr} A(i) \operatorname{tr} B(j) \right) \mathcal{L}_0 \mathcal{R}_0.$$

This is equal to $\mathcal{C}(k+1) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} c_{ij}(k+1) \mathcal{L}_i \mathcal{R}_j$ if the $c_{ij}(k+1)$ are assigned to be

$$c_{00}(k+1) = \widetilde{c}_{00}(k) - \frac{1}{k+1} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \widetilde{c}_{ij}(k) \operatorname{tr} A(i) \operatorname{tr} B(j)$$

$$c_{ij}(k+1) = \widetilde{c}_{ij}(k) \qquad \text{if } (i,j) \neq (0,0).$$

■

The analogous result for the Faddeev sequence of $\mathcal{D}$ is as follows

THEOREM 3.3. *Define the $m \times n$ coefficient matrices*

$$D(k) = (d_{ij}(k))_{i=0,j=0}^{m-1,n-1}, \qquad \textit{for } k = 0, 1, 2, \ldots, mn-1 \tag{3.17}$$

$$\widetilde{D}(k) = (\widetilde{d}_{ij}(k))_{i=0,j=0}^{m-1,n-1}, \qquad \textit{for } k = 0, 1, 2, \ldots, mn-1 \tag{3.18}$$

by the following recursive formulas:

$$D(0) = E_{11} \tag{3.19}$$

and for each $k = 0, 1, 2, \ldots, mn - 1$

$$\widetilde{d}_{00}(k) = d_{00}(k) - \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \frac{\operatorname{tr}\widetilde{A}(i)}{i+1}\frac{\operatorname{tr}\widetilde{B}(j)}{j+1} d_{ij}(k) \tag{3.20}$$

$$\widetilde{d}_{0j}(k) = d_{0j}(k) - \sum_{i=0}^{m-1} \frac{\operatorname{tr}\widetilde{A}(i)}{i+1} d_{i,j-1}(k) \qquad \textit{if } j > 0 \tag{3.21}$$

$$\widetilde{d}_{i0}(k) = d_{i0}(k) - \sum_{j=0}^{n-1} \frac{\operatorname{tr}\widetilde{B}(j)}{j+1} d_{i-1,j}(k) \qquad \textit{if } i > 0 \tag{3.22}$$

$$\widetilde{d}_{ij}(k) = d_{ij}(k) - d_{i-1,j-1}(k) \qquad \textit{if } i > 0, j > 0 \tag{3.23}$$

$$d_{00}(k+1) = \widetilde{d}_{00}(k) - \frac{1}{k+1}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \widetilde{d}_{ij}(k)\operatorname{tr} A(i)\operatorname{tr} B(j) \tag{3.24}$$

$$d_{ij}(k+1) = \widetilde{d}_{ij}(k) \qquad \textit{if } (i,j) \neq (0,0). \tag{3.25}$$

Proof. Analogous to the proof of the previous theorem, this is left to the reader. ∎

3.3. The Faddeev Sequence Formulae in Matrix-Vector Notation

The recursive equations for the Faddeev sequence of the matrix operators $\mathcal{C}$ and $\mathcal{D}$ can also be cast in matrix-vector notation. Let $p(s) = \det(sI_m - A) = s^m + p_1 s^{m-1} + \cdots + p_m$ be the characteristic polynomial of A and let $q(s) = \det(sI_n - B) = s^n + q_1 s^{n-1} + \cdots + q_n$ be the characteristic polynomial of B. Let A_c be the $m \times m$ matrix given by

$$A_c := \begin{pmatrix} -p_1 & -p_2 & \ldots & -p_{m-1} & -p_m \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \tag{3.26}$$

and let B_c be the $n \times n$ matrix given by

$$B_c := \begin{pmatrix} -q_1 & -q_2 & \dots & -q_{n-1} & -q_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \tag{3.27}$$

Furthermore let $\tau_A \in \mathbf{R}^m$ denote the vector of traces of the elements $A(0), A(1), \dots, A(m-1)$ of the Faddeev sequence of A:

$$\tau_A := \begin{pmatrix} \operatorname{tr} A(0) \\ \operatorname{tr} A(1) \\ \vdots \\ \operatorname{tr} A(m-1) \end{pmatrix} \tag{3.28}$$

and let $\tau_B \in \mathbf{R}^n$ be defined analogously. Note that for $k > 0$, $\operatorname{tr} A(k) = \operatorname{tr} \widetilde{A}(k-1) - (\operatorname{tr} \widetilde{A}(k-1)/k) \operatorname{tr} I_m = (m-k)p_k$, where the equality $p_k = -(\operatorname{tr} \widetilde{A}(k-1))/k$ is used. Of course, defining $p_0 := 1$, $\operatorname{tr} A(0) = m = mp_0$. So one can express the elements of the vector τ_A in terms of the p_k, $k = 0, 1, 2, \dots, m-1$, as follows:

$$\tau_A = \begin{pmatrix} mp_0 \\ (m-1)p_1 \\ \vdots \\ 2p_{m-2} \\ p_{m-1} \end{pmatrix} \tag{3.29}$$

An analogous formula holds for τ_B. It is now straightforward to verify that the recursive formulae given in Theorems 3.2 and 3.3 for the coefficient matrices of Faddeev sequences of $\mathcal{C}$ and $\mathcal{D}$ can be rewritten as

$$\begin{aligned} C(0) &= E_{11} \\ \widetilde{C}(0) &= A_c C(0) + C(0) B_c^T \end{aligned} \tag{3.30}$$

$$\begin{aligned} C(k) &= \widetilde{C}(k-1) - \frac{\tau_A^T \widetilde{C}(k-1) \tau_B}{k} E_{11}, \qquad k = 1, 2, \dots, mn-1 \\ \widetilde{C}(k) &= A_c C(k) + C(k) B_c^T \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} D(0) &= E_{11} \\ \widetilde{D}(0) &= D(0) - A_c D(0) B_c^T \end{aligned} \tag{3.32}$$

$$\begin{aligned} D(k) &= \widetilde{D}(k-1) - \frac{\tau_A^T \widetilde{D}(k-1)\tau_B}{k} E_{11}, \\ \widetilde{D}(k) &= D(k) - A_c D(k) B_c^T \end{aligned} \qquad k = 1, 2, \ldots, mn-1. \quad (3.33)$$

It follows that the inverse of $\mathcal{C}$ can be expressed as

$$\mathcal{C}^{-1} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \gamma_{ij} \mathcal{L}_i \mathcal{R}_j, \tag{3.34}$$

where the $m \times n$ matrix $\Gamma = (\gamma_{i-1,j-1})_{i=1,j=1}^{m,n}$ is given by the formula

$$\Gamma = \frac{mn}{\tau_A^T \widetilde{C}(mn-1)\tau_B} C(mn-1). \tag{3.35}$$

It may be interesting to note that Γ has a structure that could perhaps be called *alternating Hankel*:

$$\gamma_{ij} = -\gamma_{i-1,j+1}, \qquad \text{for } i = 1, 2, 3, \ldots, m-1; j = 0, 1, 2, \ldots, n-2.$$

This can easily be derived from the equality

$$A_c \Gamma + \Gamma B_c^T = \frac{mn}{\tau_A^T \widetilde{C}(mn-1)\tau_B} \left(A_c C(mn-1) + C(mn-1) B_c^T \right) = E_{11}$$

together with the special structure of A_c and B_c.

The inverse of $\mathcal{D}$ can be expressed as

$$\mathcal{D}^{-1} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \delta_{ij} \mathcal{L}_i \mathcal{R}_j, \tag{3.36}$$

where the $m \times n$ matrix $\Delta = (\delta_{i-1,j-1})_{i=1,j=1}^{m,n}$ is given by the formula

$$\Delta = \frac{mn}{\tau_A^T \widetilde{D}(mn-1)\tau_B} D(mn-1). \tag{3.37}$$

It may be interesting to note that Δ is a Toeplitz matrix. This can easily be derived from the equality

$$\Delta - A_c \Delta B_c^T = \frac{mn}{\tau_A^T \widetilde{D}(mn-1)\tau_B} \left(D(mn-1) - A_c D(mn-1) B_c^T \right) = E_{11}$$

together with the special structure of A_c and B_c.

3.4. Solution of the Lyapunov and Sylvester Equations

The solution of the matrix equation

$$AP + PB = K \tag{3.38}$$

can now be given by the formula

$$P = \mathcal{C}^{-1}(K) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \gamma_{ij}\mathcal{L}_i\mathcal{R}_j(K) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \gamma_{ij}A(i)KB(j). \tag{3.39}$$

Similarly the solution of the matrix equation

$$P - APB = K \tag{3.40}$$

is given by

$$P = \mathcal{D}^{-1}(K) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \delta_{ij}\mathcal{L}_i\mathcal{R}_j(K) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \delta_{ij}A(i)KB(j). \tag{3.41}$$

To write this in a concise way, the following definition will be helpful.

DEFINITION 3.4. Consider a pair (A, b) with A an $m \times m$ matrix and b an $m \times 1$ column vector. The matrix

$$F_A(b) := [A(0)b, A(1)b, \ldots, A(m-1)b] \tag{3.42}$$

is called the Faddeev reachability matrix of the pair (A, b).

The name of this matrix is clarified in the next section when relations with system theoretical concepts are treated. If $K = xy^T$ is a rank 1 matrix, where x and y are column vectors, then the solution of $AP + PB = K$ can be rewritten as

$$P = F_A(x)\Gamma\big(F_{B^T}(y)\big)^T. \tag{3.43}$$

This follows directly from (3.39) using the basic rules of matrix multiplication. Similarly the solution of $P - APB = xy^T$ is given by

$$P = F_A(x)\Delta\big(F_{B^T}(y)\big)^T. \tag{3.44}$$

Any matrix K can of course be written as a linear combination $\sum_{k=1}^{r} x_k y_k^T$ of rank 1 matrices, where $r \leq \min(m, n)$. (For instance, one can write

$K = \sum_{i=1}^{m} e_i k_i^T$ with k_i^T denoting the ith row of K.) The solution of $AP + PB = K$ is then given by

$$P = \sum_{k=1}^{r} F_A(x_k)\Gamma\big(F_{B^T}(y_k)\big)^T \tag{3.45}$$

and similarly the solution of $P - APB = K$ is then given by

$$P = \sum_{k=1}^{r} F_A(x_k)\Delta\big(F_{B^T}(y_k)\big)^T. \tag{3.46}$$

This formula has some interesting consequences, which are of interest even outside the scope of Faddeev sequence methods.

COROLLARY 3.5. *If (A, x) and (B^T, y) are pairs such that $F_A(x)$ and $F_{B^T}(y)$ are nonsingular (i.e., (A, x) and (B^T, y) are reachable pairs; cf. the next section) then:*

(i) *If Σ is the solution of the equation $A\Sigma + \Sigma B = xy^T$ then*

$$\Gamma = F_A(x)^{-1}\Sigma\big(F_{B^T}(y)^{-1}\big)^T. \tag{3.47}$$

Similarly if Σ is the solution of $\Sigma - A\Sigma B = xy^T$ then

$$\Delta = F_A(x)^{-1}\Sigma\big(F_{B^T}(y)^{-1}\big)^T. \tag{3.48}$$

(ii) *If Σ is the solution of the equation $A\Sigma + \Sigma B = xy^T$ then the solution of $AP + PB = K = \sum_{k=1}^{r} x_{(k)}y_{(k)}^T$ is equal to*

$$P = \sum_{k=1}^{r} F_A(x_{(k)})F_A(x)^{-1}\Sigma\big(F_{B^T}(y)^{-1}\big)^T\big(F_{B^T}(y_{(k)})\big)^T \tag{3.49}$$

and similarly if Σ is the solution of $\Sigma - A\Sigma B = xy^T$ then the solution of $P - APB = K = \sum_{k=1}^{r} x_{(k)}y_{(k)}^T$ is equal to

$$P = \sum_{k=1}^{r} F_A(x_{(k)})F_A(x)^{-1}\Sigma\big(F_{B^T}(y)^{-1}\big)^T\big(F_{B^T}(y_{(k)})\big)^T. \tag{3.50}$$

The second part of this corollary tells how to find a solution if the right-hand side is K in case a solution is known for a specific right-hand side xy^T such that (A, x) and (B^T, y) are reachable. This takes on an especially simple form if $K = x_{(1)}y_{(1)}^T$; then the solution P is equal to

$$P = F_A(x_{(1)})F_A(x)^{-1}\Sigma\big(F_{B^T}(y)^T\big)^{-1}\big(F_{B^T}(y_{(1)})^T\big). \tag{3.51}$$

This can also be shown directly, bypassing the Faddeev sequences of $\mathcal{C}$ and $\mathcal{D}$ as follows. Note that $A(i)$ commutes with A; therefore multiplication on the left of the equation $A\Sigma+\Sigma B = xy^T$ by $A(i)$ gives $A(A(i)\Sigma)+A(i)\Sigma B = (A(i)x)y^T$, which shows that if $x_{(1)} = A(i)x$ then $P = A(i)\Sigma$ is the solution. Taking a linear combination $x_1 = \sum_{i=0}^{m-1} \xi_{i+1}A(i)x$ one finds the solution $P = \sum_{i=0}^{m-1} \xi_{i+1}A(i)\Sigma$ of the corresponding matrix equation with right-hand side $K = x_{(1)}y^T$. The $m \times 1$ column vector $\xi = (\xi_i)_{i=1}^m$ has the property that on the one hand $\xi = F_A(x)^{-1}x_{(1)}$ and on the other hand $P = \sum_{i=1}^m \xi_i A(i-1)\Sigma$. Now considering the lth column of the matrix $\sum_{i=1}^m \xi_i A(i-1)$ one finds that it is equal to $F_A(e_l)\xi = F_A(e_l)F_A(x)^{-1}x_{(1)}$, for $l = 1, 2, \ldots, m$. To make the next step, the following remarkable lemma is required.

LEMMA 3.6. *Let $(A, x) \in \mathbf{R}^{m\times m} \times \mathbf{R}^m$ be such that $F_A(x)$ is a nonsingular $m \times m$ matrix (i.e., (A, x) is a reachable pair); then for all $u, v \in \mathbf{R}^m$ the equality*

$$F_A(u)F_A(x)^{-1}v = F_A(v)F_A(x)^{-1}u \tag{3.52}$$

holds.

Proof. Because $F_A(x)$ is nonsingular, the vectors $A(0)x, A(1)x, \ldots, A(m-1)x$ form a basis of $\mathbf{R}^m$. As $F_A(u)F_A(x)^{-1}v$ is a bilinear form in u and v, it suffices to show the equality for the cases where $u = A(k)x$ and $v = A(l)x$, with $k = 0, 1, \ldots, m-1$; $l = 0, 1, \ldots, m-1$. Because all the elements of the Faddeev sequence of A commute one has

$$\begin{aligned} F_A(A(k)x)F_A(x)^{-1}A(l)x &= A(k)F_A(x)F_A(x)^{-1}A(l)x = A(k)A(l)x \\ &= A(l)A(k)x = F_A(A(l)x)F_A(x)^{-1}A(k)x. \end{aligned}$$

■

Applying this lemma to the lth column of the matrix $\sum_{i=1}^m \xi_i A(i-1)$ one finds that it is equal to $F_A(e_l)F_A(x)^{-1}x_{(1)} = F_A(x_{(1)})F_A(x)^{-1}e_l$. Therefore the matrix is in fact equal to $F_A(x_{(1)})F_A(x)^{-1}$. A similar reasoning can be applied if y^T is replaced by $y_{(1)}^T$ in the right-hand side of the matrix equation, using the reachability of the pair (B^T, y). In this way a direct proof of the second part of the corollary above has been obtained.

REMARK. In case $m = n$ and $B = A^T$, there are several parametrized families of matrices A for which an accompanying vector b is known such that (A, b) is reachable (i.e., $F_A(b)$ nonsingular) and such that the equation $AP + PA = -bb^T$, respectively, $P - APA^T = bb^T$ has a *known* (simple) solution. For example if A stems from a balanced parametrization then

$AP + PA^T = -bb^T$ has a known *diagonal* solution matrix $P = \Sigma$ (cf., e.g., [9]). It follows that in that case the solution of $AP + PA^T = K$ can be found directly from the second part of the corollary, without going through the calculation of the Faddeev sequence of C. Another example is when A is in Schwarz form, i.e., $A = (-b_1^2/2)E_{11} + A_{sk}$, where A_{sk} is an arbitrary tridiagonal skew-symmetric matrix. Then $A + A^T = -bb^T$ if $b = b_1 e_1$, implying that the identity matrix is the solution of the Lyapunov equation. The solution of $AP+PA^T = x_{(1)}y_{(1)}^T$ is then obtained from the corollary as: $P = -F_A(x_{(1)})F_A(b)^{-1}(F_A(b)^{-1})^T(F_A(y_{(1)}))^T$. Also for the discrete-time Lyapunov equation such parametrized families are known (cf., e.g., [11]).

4. RELATIONS WITH THE CONTROLLER CANONICAL FORM OF A PAIR (A, b)

In linear systems theory (cf., e.g., [6]) a pair $(A, b) \in \mathbf{R}^{m\times m} \times \mathbf{R}^m$ is called reachable if the *reachability matrix*

$$R_A(b) := [b,\ Ab,\ A^2b, \ldots, A^{m-1}b] \tag{4.1}$$

is nonsingular. Because for the elements $A(k)$, $k = 0, 1, 2, \ldots$ of the Faddeev sequence of A the equality $A(k) = A^k + p_1A^{k-1} + \cdots + p_{k-1}A^0$ holds, the reachability matrix $R_A(b)$ is related to the matrix $F_A(b)$ of the previous section by the formula

$$R_A(b)U = F_A(b), \tag{4.2}$$

where U is the upper triangular $m \times m$ Toeplitz matrix with first row equal to $(1, p_1, p_2, \ldots, p_{m-1})$. It follows immediately that $R_A(b)$ and $F_A(b)$ have equal rank and that if one of these matrices is nonsingular then so is the other. Therefore the matrix $F_A(b)$ is called the Faddeev reachability matrix in this paper. In linear systems theory two reachable pairs $(A_{(1)}, b_{(1)})$ and $(A_{(2)}, b_{(2)})$ are considered to be equivalent if there exists a nonsingular transformation matrix T such that $A_{(2)} = TA_{(1)}T^{-1}$ and $b_{(2)} = Tb_{(1)}$. Such a transformation corresponds to a change of basis in the space $\mathbf{R}^m$ on which A operates as an endomorphism and in which b lies as a vector. This space is called the *state space*. A special choice of the state space basis can lead to a particularly simple form of the pair (A, b), which can simplify certain calculations and from which certain properties can be deduced more easily. One speaks of a canonical form for the pair (A, b). Choosing the columns of the reachability matrix as a basis for the state space leads to the well-known controllability canonical form (cf. [6, pp. 335–336]):

$$A_y = \begin{pmatrix} 0 & 0 & \dots & 0 & -p_m \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & & 1 & -p_1 \end{pmatrix} \tag{4.3}$$

$$b_y = (1, 0, \dots, 0)^T. \tag{4.4}$$

It is easy to verify that $R_{A_y}(b_y) = I_m$. Choosing the columns of the Faddeev reachability matrix as a basis for the state space leads to the well-known *controller form*. The relation of this form with the Faddeev sequence of the dynamical matrix seems not to have been noted in the literature. The controller form is denoted by (A_c, b_c):

$$A_c = \begin{pmatrix} -p_1 & -p_2 & \dots & -p_{m-1} & -p_m \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix} \tag{4.5}$$

$$b_c = (1, 0, \dots, 0)^T. \tag{4.6}$$

(Note that the notation A_c was used already in the previous section for matrices with this structure.) That this is indeed the form of the reachable pair if the columns of the Faddeev reachability matrix are chosen as basis for the state space follows simply from the fact that for $k = 0, 1, 2, \dots, m-1$,

$$A(A(k)b) = \widetilde{A}(k)b = A(k+1)b + \frac{\operatorname{tr} \widetilde{A}(k)}{k+1} b$$

and as was stated before, $\operatorname{tr} \widetilde{A}(k)/(k+1) = -p_{k+1}$. So if the columns of the Faddeev reachability matrix are chosen as the basis of the state space, then of course $b = e_1$ and $A(k-1)b = e_k$, $k = 1, 2, \dots, m$ and $A(m)b = 0$ because $A(m) = 0$. So $Ae_m = -p_m e_1$ and $Ae_k = e_{k+1} - p_k e_1$, $k = 1, \dots, m-1$ and so (A, b) is indeed in controller form. Furthermore if (A_c, b_c) is a reachable pair in controller form, then the characteristic polynomial is $\det(sI - A_c) = p(s)$ and therefore choosing the columns of the Faddeev reachability matrix as a basis of the state space keeps the reachable pair unchanged, which implies that the Faddeev reachability matrix is the identity matrix: $F_{A_c}(b_c) = I_m$. This leads to an interpretation of the coefficient matrices Γ and Δ of $\mathcal{C}^{-1}$ and $\mathcal{D}^{-1}$, respectively, as follows. Let $A = A_c$ have characteristic polynomial $p(s)$ of degree m and $B = B_c^T$ have characteristic polynomial $q(s)$ of degree n; then the equation

$$A_cP + PB_c^T = E_{11} = e_1 f_1^T$$

has solution Γ as defined in (3.35). This follows simply from (3.43) because $F_{A_c}(e_1) = I_m$ and $F_{B_c}(f_1) = I_n$. Similarly the equation

$$P - A_cPB_c^T = E_{11} = e_1 f_1^T$$

has solution Δ. Note that Γ and Δ clearly depend only on the coefficients $p_1, p_2, \ldots, p_m$; $q_1, \ldots, q_n$, which are *system invariants* and therefore Γ and Δ are themselves system invariants. If $B_c = A_c$ and A_c has its spectrum in the open left half plane then $-\Gamma$ is the positive definite *continuous-time reachability Grammian* of the pair (A_c, e_1). And if $B_c = A_c$ and A_c has its spectrum in the open unit disk then Δ is the positive definite *discrete-time reachability Grammian* of the same pair.

If $A = A_c$ and $B = B_c^T$ in the Sylvester equations studied here, a remarkable simplification of the solution formulas can be obtained, by studying the Faddeev sequence of an endomorphism in controller form A_c more closely.

First as a corollary of Lemma 3.6 one can obtain the following equality

$$A_c(i-1)e_j = A_c(j-1)e_i, \qquad i = 1, \ldots, m; \; j = 1, \ldots, m. \tag{4.7}$$

In order to derive this from Lemma 3.6, use that $F_{A_c}(e_1) = I_m$ and that $F_{A_c}(e_j)e_i = A_c(i-1)e_j$. Combining this one obtains

$$A_c(i-1)e_j = F_{A_c}(e_j)F_{A_c}(e_1)^{-1}e_i = F_{A_c}(e_i)F_{A_c}(e_1)^{-1}e_j = A_c(j-1)e_i.$$

This implies that

$$F_{A_c}(e_i) = A_c(i-1) \tag{4.8}$$

for all $i = 1, \ldots, m$.

Second, inspection of the Faddeev sequence of A_c (for small values of m—here computer algebra has turned out to be very helpful) learns that it has a remarkably simple structure! So much that one need not even calculate it; one can just apply the following theorem.

THEOREM 4.1. *Let (A_c, e_1) be in controller form and let A_c have characteristic polynomial $p(s)$ of degree m. Then the matrix $A(k)$, $k = 0, 1, 2, \ldots, m-1$ from the Faddeev sequence of $A = A_c$ can be partitioned as*

$$A(k) = \begin{bmatrix} A_1(k) \\ A_2(k) \end{bmatrix},$$

where $A_1(k)$ is $k \times m$ Toeplitz matrix (it is not there if $k = 0$) and $A_2(k)$ is an $(m-k) \times m$ Toeplitz matrix.

$A_1(k)$ is given by its first row and column: the first row is $(0, -p_{k+1}, -p_{k+2}, \ldots, -p_m, 0, \ldots, 0)$, so the number of zeroes at the end is $k-1$; the first column is zero. $A_2(k)$ is given by its last row and last column: its last row is $(0, \ldots, 0, p_0, p_1, \ldots, p_k)$, with $p_0 := 1$, the number of zeroes at the beginning is of course $m-1-k$; the last column is $(0, \ldots, 0, p_k)^T$, and the number of zeroes above p_k is $m-1-k$.

Proof. By induction. $A(0) = I_m$ is clearly of this form and

$$A(1) = A_c + p_1 I_m = \begin{pmatrix} 0 & -p_2 & \cdots & -p_{m-1} & -p_m \\ 1 & p_1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & p_1 & \vdots \\ 0 & 0 & & 1 & p_1 \end{pmatrix}$$

is also clearly of this form. Suppose (induction hypothesis) that $A(k)$ is of this form. Then the first row of $\widetilde{A}(k) = A(k)A$ is equal to $e_1^T A(k) A = (0, -p_{k+1}, -p_{k+2}, \ldots, -p_m, 0, \ldots, 0)A = (-p_{k+1}, -p_{k+2}, \ldots, -p_m, 0, \ldots, 0)$.

The other rows of $\widetilde{A}(k)$ consist of the first $m-1$ rows of $A(k)$, shifted down by one row, since we can write

$$\widetilde{A}(k) = AA(k) = e_1(-p_1, -p_2, \ldots, -p_m)A(k) + \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} A(k).$$

We find that $\widetilde{A}(k)$ can be partitioned into two Toeplitz matrices $\widetilde{A}_1(k)$ and $\widetilde{A}_2(k)$, where the first is of size $(k+1) \times m$ and the second of size $(m-k-1) \times m$. $\widetilde{A}_1(k)$ is characterized by its first row and column: the first row is $(-p_{k+1}, -p_{k+2}, \ldots, -p_m, 0, \ldots, 0)$ and the first column is $(-p_{k+1}, 0, \ldots, 0)^T$. $\widetilde{A}_2(k)$ is characterized by its last row and last column: its last row is $(0, \ldots, 0, p_0, p_1, \ldots, p_k, 0)$, with $p_0 := 1$, the number of zeroes at the beginning is $m-2-k$, and the last column is zero. Because we have that

$$A(k+1) = \widetilde{A}(k) - \frac{\operatorname{tr} \widetilde{A}(k)}{k+1} I_m = \widetilde{A}(k) + p_{k+1} I_m$$

it follows that $A(k+1)$ is obtained from $\widetilde{A}(k)$ by adding $p_{k+1} I_m$, which clearly gives the partitioning in two Toeplitz matrices as described in the theorem. ■

REMARKS. 1. From the proof of the theorem it also follows that the matrices $\widetilde{A}(k)$ allow for a similar partitioning in two Toeplitz blocks, with equally simple structure.

2. Matrices $A(k)$ and $\widetilde{A}(k)$ are so-called Sylvester matrices for particular associated pairs of polynomials (of possibly different degree).

These properties of the Faddeev sequence of A_c can be applied to simplify the solution formulas for $A_cP + PB = K$ and $P - A_cPB = K$. Let k_i^T denote the ith row of K, i.e., $K = \sum_{i=1}^{m} e_i k_i^T$. Using the fact that $F_{A_c}(e_i) = A_c(i-1)$, one obtains that the solution of $A_cP + PB = K$ is given by

$$P = \sum_{i=1}^{m} A_c(i-1)\Gamma\big(F_{B^T}(k_i)\big)^T, \tag{4.9}$$

where the matrices $A_c(i-1)$ require no calculations.

5. A SPECIAL ALGORITHM FOR THE CASE OF EQUAL CHARACTERISTIC POLYNOMIALS

Consider again the equations $AP + PB = K$ and $P - APB = K$. If the characteristic polynomials $p(s)$ and $q(s)$ of A and B are the same, then it follows from the results in the previous sections that Γ and Δ are the solutions of the equations $A_cP + PA_c^T = E_{11}$ and $P - A_cPA_c^T = E_{11}$, where A_c is in controller form and has characteristic polynomial $p(s)$. Let $\mathcal{C}_c$ denote here the linear mapping given by $P \mapsto A_cP + PA_c^T$ and similarly let $\mathcal{D}_c$ denote the mapping $P \mapsto P - A_cPA_c^T$. Then $\mathcal{C}_c$ and $\mathcal{D}_c$ have the property that they map symmetric matrices to symmetric matrices and that $\Gamma = \mathcal{C}_c^{-1}(E_{11})$ and $\Delta = \mathcal{D}_c^{-1}(E_{11})$, where obviously E_{11} is symmetric. Therefore the Faddeev sequence method can be applied to invert $\mathcal{C}_c$ and $\mathcal{D}_c$ *considered as endomorphisms of the* $\frac{1}{2}m(m+1)$*-dimensional vector space of symmetric* $m \times m$ *matrices.*

Let Σ_m denote the set of $m \times m$ symmetric matrices and let us denote the Faddeev sequence of $\mathcal{C}_{c|\Sigma_m}$ by $\{\mathcal{C}_c^{\Sigma}(k), \widetilde{\mathcal{C}}_c^{\Sigma}(k) \mid k = 0, 1, \ldots\}$ and similarly the Faddeev sequence of $\mathcal{D}_{c|\Sigma_m}$ by $\{\mathcal{D}_c^{\Sigma}(k), \widetilde{\mathcal{D}}_c^{\Sigma}(k) \mid k = 0, 1, \ldots\}$. The corresponding coefficient matrices are denoted by $C^{\Sigma}(k)$, $\widetilde{C}^{\Sigma}(k)$ (for $k = 0, 1, \ldots$) and $D^{\Sigma}(k)$, $\widetilde{D}^{\Sigma}(k)$ (for $k = 0, 1, \ldots$), respectively. To calculate these, one needs to know the formula for the trace of a linear combination of the linear matrix operators $(\frac{1}{2}\mathcal{L}_k\mathcal{R}_l + \frac{1}{2}\mathcal{L}_l\mathcal{R}_k)|_{\Sigma_m}$, where $\mathcal{L}_k$ stands for multiplication on the left by $A_c(k)$ and $\mathcal{R}_l$ stands for multiplication on the right by $A_c^T(l)$, where $k, l \in \{0, 1, 2, \ldots, m-1\}$. This is given in the following lemma.

LEMMA 5.1. *The trace of the linear matrix operator* $(\frac{1}{2}\mathcal{L}_k\mathcal{R}_l+\frac{1}{2}\mathcal{L}_l\mathcal{R}_k)|_{\Sigma_m}$ *considered as an endomorphism on the set* Σ_m *of* $m \times m$ *matrices is equal to* $\frac{1}{2}\mathrm{tr}\, A_\mathrm{c}(k)\mathrm{tr}\, A_\mathrm{c}(l) + \frac{1}{2}\mathrm{tr}\,\{A_\mathrm{c}(k)A_\mathrm{c}(l)\}$.

Proof. Choose the usual matrix inner product $\langle P, Q\rangle = \mathrm{tr}\,\{P^TQ\}$ as inner product on Σ_m. (Of course the choice of inner product does not affect the value of the trace.) An orthonormal basis with respect to this inner product is given by

$$\{E_{ii}\}_{i=1}^m \cup \left\{ \frac{E_{ij}+E_{ji}}{\sqrt{2}} \,\middle|\, i > j \right\}.$$

It contains $\frac{1}{2}m(m+1)$ elements, all of which are of course symmetric. The trace of $(\frac{1}{2}\mathcal{L}_k\mathcal{R}_l + \frac{1}{2}\mathcal{L}_l\mathcal{R}_k)|_{\Sigma_m}$ is now given by

$$\begin{aligned}
&\sum_{i=1}^m \left\langle E_{ii},\ \left(\frac{1}{2}\mathcal{L}_k\mathcal{R}_l + \frac{1}{2}\mathcal{L}_l\mathcal{R}_k\right)(E_{ii})\right\rangle \\
&\quad + \sum_{i>j}\left\langle \frac{E_{ij}+E_{ji}}{\sqrt{2}},\ \left(\frac{1}{2}\mathcal{L}_k\mathcal{R}_l + \frac{1}{2}\mathcal{L}_l\mathcal{R}_k\right)\left(\frac{E_{ij}+E_{ji}}{\sqrt{2}}\right)\right\rangle \\
&\quad = \frac{1}{2}\sum_{i=1}^m \mathrm{tr}\big\{E_{ii}A_\mathrm{c}(k)E_{ii}A_\mathrm{c}^T(l)\big\} + \frac{1}{2}\sum_{i=1}^m \mathrm{tr}\big\{E_{ii}A_\mathrm{c}(l)E_{ii}A_\mathrm{c}^T(k)\big\} \\
&\qquad + \frac{1}{4}\sum_{i>j}\mathrm{tr}\big\{E_{ij}A_\mathrm{c}(k)E_{ij}A_\mathrm{c}^T(l) + E_{ij}A_\mathrm{c}(k)E_{ji}A_\mathrm{c}^T(l) \\
&\qquad\qquad + E_{ji}A_\mathrm{c}(k)E_{ij}A_\mathrm{c}^T(l) + E_{ji}A_\mathrm{c}(k)E_{ji}A_\mathrm{c}^T(l)\big\} \\
&\qquad + \frac{1}{4}\sum_{i>j}\mathrm{tr}\big\{E_{ij}A_\mathrm{c}(l)E_{ij}A_\mathrm{c}^T(k) + E_{ij}A_\mathrm{c}(l)E_{ji}A_\mathrm{c}^T(k) \\
&\qquad\qquad + E_{ji}A_\mathrm{c}(l)E_{ij}A_\mathrm{c}^T(k) + E_{ji}A_\mathrm{c}(l)E_{ji}A_\mathrm{c}^T(k)\big\} \\
&\quad = \sum_{i=1}^m e_i^TA_\mathrm{c}(k)e_ie_i^TA_\mathrm{c}^T(l)e_i \\
&\qquad + \frac{1}{2}\sum_{i>j}\big\{e_j^TA_\mathrm{c}(k)e_ie_j^TA_\mathrm{c}(l)^Te_i + e_j^TA_\mathrm{c}(k)e_je_i^TA_\mathrm{c}(l)^Te_i \\
&\qquad\qquad + e_i^TA_\mathrm{c}(k)e_ie_j^TA_\mathrm{c}(l)^Te_j + e_i^TA_\mathrm{c}(k)e_je_i^TA_\mathrm{c}(l)^Te_j\big\}.
\end{aligned}$$

Working this out shows that this is equal to

$$\frac{1}{2}\sum_{ij} e_i^TA_\mathrm{c}(k)e_ie_j^TA_\mathrm{c}^T(l)e_j + \frac{1}{2}\sum_{ij} e_i^TA_\mathrm{c}(k)e_je_i^TA_\mathrm{c}(l)^Te_j,$$

which is clearly equal to

$$\tfrac{1}{2}\mathrm{tr}\, A_c(k)\mathrm{tr}\, A_c(l) + \tfrac{1}{2}\mathrm{tr}\,\{A_c(k)A_c(l)\}.$$

■

The sequence of coefficient matrices $C^{\Sigma}(k)$ is now given by

$$\begin{aligned} C^{\Sigma}(0) &= E_{11} \\ \widetilde{C}^{\Sigma}(0) &= A_c C^{\Sigma}(0) + C^{\Sigma}(0)A_c^T \end{aligned} \tag{5.1}$$

$$\begin{aligned} C^{\Sigma}(k) &= \widetilde{C}^{\Sigma}(k-1) - \frac{\tau_A^T \widetilde{C}^{\Sigma}(k-1)\tau_A + \langle T_A, \widetilde{C}^{\Sigma}(k-1)\rangle}{2k} E_{11} \\ \widetilde{C}^{\Sigma}(k) &= A_c C^{\Sigma}(k) + C^{\Sigma}(k)A_c^T, \\ & k = 1, 2, \ldots, \tfrac{1}{2}m(m+1) - 1 \end{aligned} \tag{5.2}$$

and Γ is obtained by using the formula

$$\begin{aligned} \Gamma = {} & \frac{m(m+1)}{\tau_A^T \widetilde{C}^{\Sigma}\big(\tfrac{1}{2}m(m+1)-1\big)\tau_A + \big\langle T_A, \widetilde{C}^{\Sigma}\big(\tfrac{1}{2}m(m+1)-1\big)\big\rangle} \\ & \times C^{\Sigma}\big(\tfrac{1}{2}m(m+1)-1\big), \end{aligned} \tag{5.3}$$

where $\langle X, Y\rangle = \Sigma_{ij} X_{ij} Y_{ij}$ denotes the inner product of two matrices X and Y of the same size and where T_A denotes the $m \times m$ matrix which has $\mathrm{tr}\,\{A_c(k-1)A_c(l-1)\}$ as its (k,l)th element. Similarly the sequence of coefficient matrices $D^{\Sigma}(k)$ is given recursively by

$$\begin{aligned} D^{\Sigma}(0) &= E_{11} \\ \widetilde{D}^{\Sigma}(0) &= D^{\Sigma}(0) - A_c D^{\Sigma}(0)A_c^T \end{aligned} \tag{5.4}$$

$$\begin{aligned} D^{\Sigma}(k) &= \widetilde{D}^{\Sigma}(k-1) - \frac{\tau_A^T \widetilde{D}^{\Sigma}(k-1)\tau_A + \langle T_A, \widetilde{D}^{\Sigma}(k-1)\rangle}{2k} E_{11} \\ \widetilde{D}^{\Sigma}(k) &= D^{\Sigma}(k) - A_c D^{\Sigma}(k)A_c^T, \\ & k = 1, 2, \ldots, \tfrac{1}{2}m(m+1) - 1 \end{aligned} \tag{5.5}$$

and Δ is obtained from

$$\begin{aligned} \Delta = {} & \frac{m(m+1)}{\tau_A^T \widetilde{D}^{\Sigma}\big(\tfrac{1}{2}m(m+1)-1\big)\tau_A + \big\langle T_A, \widetilde{D}^{\Sigma}\big(\tfrac{1}{2}m(m+1)-1\big)\big\rangle} \\ & \times D^{\Sigma}\big(\tfrac{1}{2}m(m+1)-1\big). \end{aligned} \tag{5.6}$$

Both algorithms require just $\frac{1}{2}m(m+1)$ iterations to obtain Γ and Δ instead of the previous m^2 iterations. The calculation of the traces of the operators is more involved though, but does not increase the overall complexity of the algorithms.

6. EXAMPLES

In this section some outcomes of the algorithm are presented. In fact we give the formula for Γ in the case $p(s) = q(s)$ in terms of the coefficients of $p(s)$ for several values of m. One reason for doing this, apart from showing what kind of results can be obtained with this algorithm, is that by substitution of $p_k = -(\operatorname{tr} \widetilde{A}(k-1))/k$ one can obtain the matrix Γ for arbitrary parametrizations of A, and using our formulas, one can relatively easily obtain the solution of the Sylvester or Lyapunov equation involved using Γ. Because Γ is alternating Hankel it suffices to give a common denominator, together with the elements of the first row and last column. Furthermore, because it is as well symmetric, each element of the matrix for which $i + j$ is odd is zero.

The formulae presented below all apply to the continuous-time case, because experience shows that in the discrete-time case the outcomes are usually more involved. They have been calculated on a 486-based personal computer. Once the formulae are available, it is possible within the Maple and Mathematica software packages to substitute numerical values for the coefficients and then to obtain the outcomes with prespecified numerical accuracy.

For $m = 4$ and A given by

$$A = \begin{bmatrix} -p_1 & -p_2 & -p_3 & -p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{6.1}$$

the coefficient matrix Γ satisfying $A\Gamma + \Gamma A^T = E_{11}$ is calculated as

$$\Gamma = \begin{bmatrix} r_1 & 0 & r_2 & 0 \\ 0 & -r_2 & 0 & -r_3 \\ r_2 & 0 & r_3 & 0 \\ 0 & -r_3 & 0 & -r_4 \end{bmatrix}, \tag{6.2}$$

where

$$r_1 = \tfrac{1}{2}(-p_2 p_3 + p_1 p_4)/d \tag{6.3}$$

$$r_2 = \tfrac{1}{2} p_3/d \tag{6.4}$$

$$r_3 = -\tfrac{1}{2} p_1/d \tag{6.5}$$

$$r_4 = \tfrac{1}{2}(p_1 p_2 - p_3)/(p_4 d) \tag{6.6}$$

and where the polynomial d appearing in the denominators of $r_1, \dots, r_4$ is given by

$$d = p_1 p_2 p_3 - p_3^2 - p_1^2 p_4. \tag{6.7}$$

These formulae were obtained with Mathematica in about 15 sec, using the general algorithm that does not exploit the available symmetry. The direct approach with Kronecker products using the Mathematica routines LinearSolve and Factor (to obtain simplified results) took about 15 sec as well. (However, in the discrete-time case the direct approach needed 5 min to obtain the solution and many more minutes for simplification, while the algorithm of this paper was capable of yielding the simplified results in less than 2 min.)

For $m = 7$ and A given by

$$A = \begin{bmatrix} -p_1 & -p_2 & -p_3 & -p_4 & -p_5 & -p_6 & -p_7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{6.8}$$

the coefficient matrix Γ satisfying $A\Gamma + \Gamma A^T = E_{11}$ is calculated as

$$\Gamma = \begin{bmatrix} r_1 & 0 & r_2 & 0 & r_3 & 0 & r_4 \\ 0 & -r_2 & 0 & -r_3 & 0 & -r_4 & 0 \\ r_2 & 0 & r_3 & 0 & r_4 & 0 & r_5 \\ 0 & -r_3 & 0 & -r_4 & 0 & -r_5 & 0 \\ r_3 & 0 & r_4 & 0 & r_5 & 0 & r_6 \\ 0 & -r_4 & 0 & -r_5 & 0 & -r_6 & 0 \\ r_4 & 0 & r_5 & 0 & r_6 & 0 & r_7 \end{bmatrix}, \tag{6.9}$$

where

$$\begin{aligned} r_1 = -\tfrac{1}{2}\big(& - p_1 p_4^3 p_7 - p_2^2 p_4 p_5 p_7 + p_1 p_4^2 p_6 p_5 + p_4^2 p_7 p_5 + p_6 p_7^2 \\ & + p_6 p_2^2 p_5^2 + p_2^3 p_7^2 - p_4 p_6 p_5 p_2 p_3 - 2 p_6 p_2^2 p_7 p_3 + p_4^2 p_7 p_2 p_3 \\ & - p_1 p_4 p_6^2 p_3 + p_6^2 p_5 p_3 - p_6 p_4 p_5^2 + p_1^2 p_6^3 - 2 p_7 p_1 p_6^2 - 2 p_7^2 p_4 p_2 \\ & + 3 p_7 p_1 p_6 p_4 p_2 - 2 p_1 p_5 p_6^2 p_2 + p_5 p_6 p_7 p_2 + p_6^2 p_2 p_3^2 \big)/d \end{aligned} \tag{6.10}$$

$$\begin{aligned}
r_2 &= \tfrac{1}{2}\big(p_6^2p_3^2 + p_7^2p_2^2 + p_5p_6p_7 - 2p_3p_7p_2p_6 + p_5^2p_2p_6 - p_1p_5p_6^2 \\
&\qquad - p_7^2p_4 - p_7p_5p_2p_4 + p_7p_1p_6p_4 - p_5p_6p_3p_4 + p_7p_4^2p_3\big)/d \qquad (6.11)\\
r_3 &= \tfrac{1}{2}\big(-p_2p_7^2 - p_1p_4^2p_7 + p_4p_7p_5 + p_1p_4p_6p_5 - p_6p_5^2 \\
&\qquad + p_6p_7p_3 - p_1p_6^2p_3 + p_6p_7p_1p_2\big)/d \qquad (6.12)\\
r_4 &= \tfrac{1}{2}\big(p_7^2 + p_6p_5p_3 - p_7p_3p_4 - 2p_6p_7p_1 - p_6p_2p_5p_1 \\
&\qquad + p_7p_2p_4p_1 + p_6^2p_1^2\big)/d \qquad (6.13)\\
r_5 &= -\tfrac{1}{2}\big(p_6p_3^2 - p_2p_3p_7 + p_5p_7 + p_2^2p_7p_1 - p_2p_3p_6p_1 \\
&\qquad - p_5p_6p_1 - p_7p_1p_4 + p_6p_1^2p_4\big)/d \qquad (6.14)\\
r_6 &= -\tfrac{1}{2}\big(-p_5^2 + 2p_1p_5p_4 - p_1^2p_4^2 - p_1p_6p_3 + p_7p_3 - p_3^2p_4 \\
&\qquad + p_1^2p_6p_2 - p_1p_7p_2 + p_5p_3p_2 + p_1p_4p_3p_2 - p_1p_5p_2^2\big)/d \qquad (6.15)\\
r_7 &= -\tfrac{1}{2}\big(-p_5^2p_3p_2 + p_7p_3^2p_2 + p_1^3p_6^2 + p_1p_5^2p_2^2 + p_1p_5p_7p_2 + p_1^2p_4p_7p_2 \\
&\qquad - 2p_1^2p_5p_6p_2 - p_1p_5p_4p_3p_2 + p_1p_6p_3^2p_2 - p_7p_1p_3p_2^2 + p_4p_5p_3^2 \\
&\qquad + p_5^3 - p_6p_1^2p_4p_3 - 2p_5p_7p_3 + p_1^2p_4^2p_5 - 2p_1p_4p_5^2 + p_1p_7^2 \\
&\qquad - 2p_1^2p_7p_6 - p_6p_3^3 + 3p_5p_6p_1p_3\big)/(dp_7) \qquad (6.16)
\end{aligned}$$

and where the polynomial d appearing in the denominators of $r_1, \ldots, r_7$ is given by

$$\begin{aligned}
d = {} & -p_7^2p_3p_2^2 + 2p_7p_1p_5p_4^2 + p_7p_3p_1p_2p_4^2 - p_3^2p_7p_4^2 + p_3p_7p_5p_2p_4 \\
& - p_7p_3p_1p_6p_4 + 3p_1^2p_7p_2p_6p_4 + 2p_7^2p_3p_4 - p_7p_5^2p_4 + p_1p_2p_3^2p_6^2 \\
& - p_1^2p_7p_4^3 - p_2p_3p_5^2p_6 - 2p_1^2p_2p_5p_6^2 - p_1^2p_3p_4p_6^2 + p_1^2p_4^2p_5p_6 \\
& - p_1p_2p_3p_4p_5p_6 + 3p_1p_3p_5p_6^2 - 3p_1^2p_7p_6^2 + p_5^3p_6 + p_1^3p_6^3 - p_3^3p_6^2 \\
& - p_7p_1p_5p_2^2p_4 - 3p_1p_7^2p_2p_4 - 2p_4p_1p_5^2p_6 + p_7p_1p_5p_2p_6 \\
& - 3p_3p_7p_5p_6 - 2p_7p_3p_1p_2^2p_6 + 2p_3^2p_7p_2p_6 + p_1p_5^2p_2^2p_6 + p_1p_7^2p_2^3 - p_7^3 \\
& + p_5p_7^2p_2 + 3p_1p_7^2p_6 + p_3^2p_4p_5p_6. \qquad (6.17)
\end{aligned}$$

These formulae were obtained using Maple in somewhat less than 1 hr, again using the general algorithm that does not exploit the symmetry. The direct approach using Kronecker products turned out to be infeasible for this particular case $m = 7$, because it amounts to solving a linear system of equations of size 49×49. As one may note, the formulae for the case $m = 4$ can be reobtained by restricting to the 4×4 left-upper block of Γ, substituting $p_5 = p_6 = p_7 = x$, cancelling common factors x, and then setting x to zero.

7. DISCUSSION, SUMMARY, AND FURTHER RESEARCH

The algorithms for solving Lyapunov and Sylvester equations developed in this paper have turned out to be useful for symbolic calculations. As the dimensions of the problem increase, the Faddeev-based recursive algorithms increasingly outperform a direct approach algorithm using Kronecker products. However, modifications that substantially reduce the number of multiplications involved are possible. Also alternative methods that exhibit comparable performance have been described in the literature (although they were not suggested specifically as algorithms for symbolic calculations); see e.g. [8, 12] and the references given there. A more detailed analysis of numerical complexity (defined as the number of multiplications involved) is given below.

If the algorithms are to be used for numerical calculation, care should be taken because numerical round-off errors tend to rapidly destroy the accuracy of final and intermediate results. (On a computer with 16-digit accuracy, the outcomes are generally unreliable for m or n larger than 4.) However, if A and B^T are given in controller form, the analytic structure of the Faddeev sequences of A and B is known, which may help to improve on numerical performance of the algorithms. Symbolic formulae obtained with the algorithms (such as given in the previous section for $m = 4$ and $m = 7$) could also be taken as a starting point, thus bypassing the numerically most ill-conditioned computations.

However, the main reason for working out the numerical complexity is that to the best of our knowledge there are no simple rules for calculation of the symbolic complexity and the numerical complexity may give some partial insight into the complexity of the corresponding symbolic algorithm.

If $m = n$, the numerical complexity of the Faddeev sequence-based algorithms in this paper is $8n^4 + O(n^3)$. This amount is obtained by adding the numerical complexities of the various stages of the algorithms.

(i) The Faddeev sequence calculation of an $n \times n$ matrix requires $n - 1$ matrix multiplications and is therefore dominated by a term n^4. In case of a Sylvester equation two such sequences are needed.

(ii) The Faddeev sequence algorithm for the Sylvester operator involves n^2 iterations, each iteration containing two matrix multiplications for which the companion structure can be exploited (two times n^2 multiplications) and the calculation of the trace of the operator ($n^2 + n$ multiplications). Thus, the Faddeev sequence calculations for the Sylvester operator involve $3n^4 + O(n^3)$ multiplications.

(iii) Decomposition of a general right-hand side K of full rank n into n rank 1 matrices is immediate. From formulae (3.45) and (3.46) it is seen that $2n$ Faddeev reachability matrices are required. If the decomposition of K into products of unit basis vectors e_i and rows k_i^T is taken, n of these Faddeev reachability matrices consist of selected columns from the matrices in the Faddeev sequence of A only and are obtained without further effort. The remaining n Faddeev reachability matrices are obtained from the Faddeev sequence of B using n^4 multiplications.

(iv) To finally compute P from (3.45) and (3.46) $2n$ matrix multiplications are required, needing $2n^4$ multiplications. Thus the total count becomes $8n^4 + O(n^3)$ as claimed above.

As has been noted in [8], it is possible to calculate matrices Γ and Δ in an alternative way by solving an easily constructed linear system of equations of size $n \times n$. This requires $n^3 + O(n^2)$ multiplications only. Therefore, stage (ii) above can be bypassed in a way that reduces the total numerical complexity to $5n^4 + O(n^3)$. It shows the inefficiency in terms of the number of multiplications of the Faddeev recursions with respect to the Sylvester operators but leaves the interpretation of the algorithms intact.

Another observation that can be made is the following. If the coefficients of the characteristic polynomials of A and B are known, the Faddeev recursions can easily be modified to directly generate the columns of the Faddeev reachability matrices. One would just have to multiply Eqs. (2.1) and (2.2) by the vectors x_k required in (3.45) and (3.46) and make use of relationship (2.3). In this way, stages (i) and (iii) are combined into one procedure involving n^4 multiplications for matrix A and its Faddeev reachability matrices and also n^4 for B, etc. The dominating term in the total count for stages (i) and (iii) is thus reduced from $3n^4$ to $2n^4$. A numerical complexity for the general algorithm of $4n^4 + O(n^3)$ is then obtained. In the present algorithm the coefficients of the characteristic polynomials are computed from the Faddeev recursion, which requires n^4 multiplications. However, there are ways of obtaining these coefficients by alternative means in just $O(n^3)$ operations; we do not go into that here.

We now comment upon some alternative approaches described in the literature. In [12] a different procedure that explicitly addresses a situation equivalent to the case where A and B^T are in controllability form (still with $K = E_{11}$) has been described. As stated there, the numerical complexity of that method is $\frac{17}{3}n^4 + O(n^3)$, which is slightly inferior to the complexity obtained above.

The results of [8] can be summarized as follows. First a general Sylvester equation is reduced to an equation of special type, involving the controllability form. Next it is transformed to an equation involving the controller form and finally this controller form equation is efficiently handled by solving a linear system of size $n \times n$. Formulae similar to (3.45) and (3.46), also appearing in [12], are used to arrive at an expression for the solution to the original equation. This method is less direct than the Faddeev sequence-based method of this paper in that a detour is taken via the controllability form, which, however, costs only $O(n^3)$ extra multiplications. Its numerical complexity therefore is similar to that of the present algorithms (after the proposed modifications): $4n^4 + O(n^3)$.

About the complexity and memory requirements of the algorithm for symbolic computation little can be said in general. Here, complexity is mainly determined by the representation of all intermediate quantities in the algorithm and therefore explicitly depends on the parametrization of A, B, and K. However, we feel there is good reason to believe that when A and B are on controller form, the Faddeev sequence-based formulae of this paper are well suited for symbolically solving Sylvester equations. One reason for this is the simple analytic structure of the matrices in the Faddeev sequences described in Theorem 4.1, something that does not occur for powers of companion matrices (which appear when using the controllability form).

The matrix-algebra approach of this paper in conjunction with Faddeev's algorithm has turned out to be a powerful instrument with which to gain more theoretical insight into the structure and the solution of Lyapunov and Sylvester equations. In particular the role of the Faddeev reachability matrix has been commented upon, as it provides the basis for the state space of a linear SISO system that leads to its controller canonical form. It also becomes manifest in the formulae that specify how the solution to Lyapunov and Sylvester equations can be quickly obtained for an arbitrary right-hand side K provided a solution is available for a right-hand side xy^T of rank 1 with (A, x) and (B^T, y) reachable pairs. Such situations are quite common when working with balanced realizations, both in the continuous-time and discrete-time case.

Important simplifications have been shown to occur when A and B have the same characteristic polynomial $p(s) = q(s)$, of which the symmetric (Lyapunov) case where $B = A^T$ is an example. An alternative algorithm for the Faddeev recursion of the Sylvester operator has been developed for this case, which requires only $\frac{1}{2}n(n+1)$ iterations instead of the usual n^2.

The situation where A and B^T are in controller form and $K = E_{11}$ has been recognized as the central issue to be further investigated. Several alternative approaches for obtaining Γ and Δ, which exploit their alternating Hankel and Toeplitz structure, are currently being studied. Other topics under research include the symbolic calculation of various Riemannian metrics on spaces of linear systems (which are calculated by repeated solution of a number of Lyapunov equations; cf. [4, 10]), including the Fisher information metric for stable linear SISO systems driven by Gaussian white noise. Further subjects of interest are the generalization of the concept of a Faddeev reachability matrix to the multivariate case and the extension of the methods in this paper to linear matrix equations of the more general form $APB + CPD = K$.

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